Derived rules

When describing the proof rule modus tollens (MT), we mentioned that it is not a primitive rule of natural deduction, but can be derived from some of the other rules. Here is the derivation of

$$\frac{\varphi \rightarrow \psi \quad \neg\psi}{\neg\varphi}\ \text{MT}$$

from →e, ¬e and ¬i:

1 φ → ψ premise

2 ¬ψ premise

3 φ assumption

4 ψ →e 1, 3

5 ⊥ ¬e 4, 2

6 ¬φ ¬i 3−5

We could now go back through the proofs in this chapter and replace applications of MT by this combination of →e, ¬e and ¬i. However, it is convenient to think of MT as a shorthand (or a macro).

The same holds for the rule e

$$\frac{\varphi}{\neg\neg\varphi}\ \neg\neg\text{i}$$

It can be derived from the rules ¬i and ¬e, as follows: 1 φ premise 2 ¬φ assumption 3 ⊥ ¬e 1, 2 4 ¬¬φ ¬i 2−3 There are (unboundedly) many such derived rules which we could write down. However, there is no point in making our calculus fat and unwieldy; and some purists would say that we should stick to a minimum set of rules, all of which are independent of each other. We don't take such a purist view. Indeed, the two derived rules we now introduce are extremely useful. You will find that they crop up frequently when doing exercises in natural deduction, so it is worth giving them names as derived rules. In the case of the second one, its derivation from the primitive proof rules is not very obvious. The first one has the Latin name reductio ad absurdum. It means 'reduction to absurdity' and we will simply call it proof by contradiction (PBC for short). The rule says: if from ¬φ we obtain a contradiction, then we are entitled to deduce φ:

¬φ

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$$\frac{\bot}{\varphi}\ \text{PBC}$$

This rule looks rather similar to ¬i, except that the negation is in a different place. This is the clue to how to derive PBC from our basic proof rules. Suppose we have a proof of ⊥ from ¬φ. By →i, we can transform this into a proof of ¬φ → ⊥ and proceed as follows:

1 ¬φ → ⊥ given

2 ¬φ assumption

3 ⊥ →e 1, 2

4 ¬¬φ ¬i 2−3

5 φ ¬¬e 4

This shows that PBC can be derived from →i, ¬i, →e and ¬¬e. The final derived rule we consider in this section is arguably the most useful to use in proofs, because its derivation is rather long and complicated, so its usage often saves time and effort. It also has a Latin name, tertium non datur ; the English name is the law of the excluded middle, or LEM for short. It simply says that φ ∨ ¬φ is true: whatever φ is, it must be either true or false; in the latter case, ¬φ is true. There is no third possibility (hence excluded middle): the sequent φ ∨ ¬φ is valid. Its validity is implicit, for example, whenever you write an if-statement in a programming language: 'if B {C1} else {C2}' relies on the fact that B ∨ ¬B is always true (and that B and ¬B can never be true at the same time). Here is a proof in natural deduction that derives the law of the excluded middle from basic proof rules:

1 ¬(φ ∨ ¬φ) assumption

2 φ assumption

3 φ ∨ ¬φ ∨i1 2

4 ⊥ ¬e 3, 1

5 ¬φ ¬i 2−4

6 φ ∨ ¬φ ∨i2 5

7 ⊥ ¬e 6, 1

8 ¬¬(φ ∨ ¬φ) ¬i 1−7

9 φ ∨ ¬φ ¬¬e 8